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APPLICATION OF THE FINITE ELEMENT METHOD TO CONVECTION HEAT TRANSFER BETWEEN PARALLEL PLANES

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Abstract—The basis of the finite element method of solution to a partial differential equation, and the associated numerical procedure, are outlined. The method is then applied to a convection heat transfer problem. This problem consists of determining the temperature distribution and the axial variation of the local Nusselt number for a fluid of constant physical properties flowing between two infinite parallel planes, at which surface temperatures or heat fluxes are specified. The flow is hydrodynamically developed. The validity of the method is verified and some aspects of the method discussed.

NOMENCLATURE

A,	area;
a, b, c,	functions of nodal co-ordinates;
C,	longitudinal surface temperature gradient;
C_p,	specific heat at constant pressure of the fluid;
D_h,	hydraulic diameter, $4R$;
h,	convective heat transfer coefficient;
$[h]$,	a matrix defined by equation (19);
I, I',	minimization functionals;
i, j, m,	nodes;
k,	thermal conductivity of the fluid;
l,	length;
N,	shape function, defined by equation (16);
Nu,	Nusselt number, hD_h/k;
n,	normal to surface;
$\{P\}$,	a matrix defined by equation (23);
Pe,	Péclét number, $RePr$;
Pr,	Prandtl number, $C_p\mu/k$;
Q,	rate of internal heat generation;
q,	heat flux;
R,	half plate spacing;
Re,	Reynolds number, $u_{max} D_h/\mu$;
$\boldsymbol{r}$,	position vector;
S,	surface;
s,	distance along surface S;
T,	temperature;
T_∞,	ambient temperature;
t,	time;
u,	unknown function;
u,	x-component of velocity;
u^+,	dimensionless velocity, u/u_{max};
V,	volume;
$\boldsymbol{v}$,	velocity;
X, Y,	dimensionless cartesian co-ordinates;
x, y,	dimensional cartesian co-ordinates;
[ ],	matrix;
{ },	column vector.

Greek symbols

ρ,	density of the fluid;
Δ,	area of triangular element;
μ,	absolute viscosity of the fluid;
θ,	dimensionless temperature $(T - T_e)/\varepsilon$;
ε,	temperature non-dimensionalisation factor;
δ_{fi},	$\begin{cases} 1 \text{ if } f = i \\ 0 \text{ if } f \neq i. \end{cases}$

Subscripts

w,	wall;
e,	entrance section or element;

$i, j, m,$ nodes;
$mm,$ mixed mean;
$s,$ surface.

INTRODUCTION

It is well recognized that a physical problem governed by a set of differential equations may be equivalently expressed as an extremum problem by the methods of the calculus of variations [1, 2]. Thus physical problems governed by the two-dimensional Poisson equation

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + Q = 0 \tag{1}$$

subject to the boundary conditions

$$T = T_s \quad \text{on } S_T$$

$$-k\frac{\partial T}{\partial n} = q \quad \text{on } S_q$$

$$-k\frac{\partial T}{\partial n} = h(T - T_\infty) \text{ on } S_h \tag{2}$$

may be solved by minimising the following integral or functional

$$I = \int_A \left[\frac{1}{2}\left\{k_x\left(\frac{\partial T}{\partial x}\right)^2 + k_y\left(\frac{\partial T}{\partial y}\right)^2\right\} - QT\right] \mathrm{d}A + \int_{S_q} qT\,\mathrm{d}S + \int_{S_h} (\tfrac{1}{2}hT^2 - hTT_\infty)\,\mathrm{d}S \tag{3}$$

with respect to the unknown function T. By variational methods (in particular, Euler's theorem), it may be shown that the satisfaction of equation (1) is a necessary and sufficient condition to render I stationary.

Physical problems of the Poisson equation type have been solved numerically by the finite element method. An excellent description of the numerical procedure for these and other problems is given in [3].

Although the equivalent functional for problems involving transport phenomena are well-known, few attempts seem to have been made to apply the finite element method to such problems. In this paper, a convection heat transfer problem is solved numerically by the finite element method. The purpose of the paper is to draw attention to the power and flexibility of the method.

The problem considered is that of a fluid of constant properties flowing in steady, laminar motion between two infinite stationary parallel planes. There is no internal heat generation and viscous dissipation is neglected. The velocity profile is fully developed and hence parabolic (see Fig. 1). The temperature of the fluid upstream

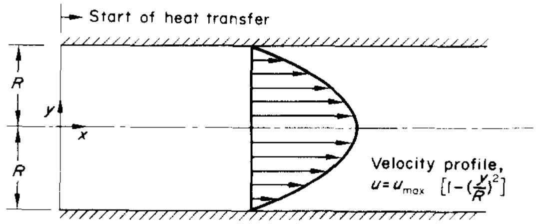

FIG. 1. Convective heat transfer problem.

of the origin is uniform across the section at T_e. Downstream of the origin, heat transfer to the fluid occurs. The problem is to determine the temperature distribution and the variation of the local Nusselt number along the direction of the flow.

A variety of boundary conditions can be imposed at the two surfaces. It has been shown [4] that, because of the linearity of the governing equation, any arbitrary surface temperature or surface heat flux boundary condition can be satisfied by superimposing solutions of two or more of the four fundamental problems:

A. One surface at a constant temperature different from that of the entering fluid, the other surface at the same temperature as the entering fluid;
B. One surface with a constant heat flux, the other surface insulated;
C. One surface at a constant temperature

different from that of the entering fluid, the other surface insulated;

D. One surface with a constant heat flux, the other surface at the temperature of the entering fluid.

All the fundamental solutions, together with those for both surface temperatures constant and both linearly varying, have been obtained using the finite element method. However, for the purpose of illustrating the application of the method to problems of this kind, only the results for the following boundary conditions will be presented:

1. Both surface temperatures linearly varying $T_w = T_e + CX$
2. One surface at constant heat flux q_w, the other surface insulated.

Theoretical solutions to these problems of heat transfer between parallel planes exist; they will be compared with the numerical solutions obtained by the finite element method.

SOLUTION BY THE FINITE ELEMENT METHOD

In general terms, the problem facing us is that of minimising a functional $I(u)$ which consists of surface and volume integrals over the continuum space, with respect to an unknown function $u(x, y, z)$. In order to perform the minimization numerically, the problem region is divided into "finite elements" by imaginary surfaces. In three-dimensional problems these "finite elements" may take the form of tetrahedrons, octahedrons, etc, and in two-dimensional problems they may be triangles, quadrilaterals or some other geometrical shape (see Fig. 2). The unknown function u is then defined in terms of the nodal values of u. For a triangular element with nodes i, j, m, for example, the unknown function within the element is defined by

$$u = N_i u_i + N_j u_j + N_m u_m \tag{4}$$

where u_i, u_j, u_m, which are to be found, are the values of u at nodes i, j, m respectively, and N_i, N_j, N_m are known "shape" functions of the nodal co-ordinates which must ensure the uniqueness and continuity of u. The equivalent functional is then a function of the nodal values u_i which can now be obtained by minimising I with respect to u_i, that is by setting

$$\frac{\partial I}{\partial u_i} = 0$$

for all i.

The functional I may be expressed as a summation of elemental values of I, denoted by I^e, that is

$$I = \sum_e I^e. \tag{5}$$

Equation (5) is true if u and all its derivatives up to the $(n\text{-}1)$th order are continuous and finite at the interface between adjacent elements, where n is the highest order of the derivative of u appearing in I.

Hence, the solution is obtained by setting

$$\sum_e \frac{\partial I^e}{\partial u_i} = 0 \tag{6}$$

for all i. Only elements for which i is a node need be considered in the summation. For all other elements, I^e is independent of u_i. Equation (6) will yield one equation for each node, giving rise to a set of simultaneous equations in u_i equal

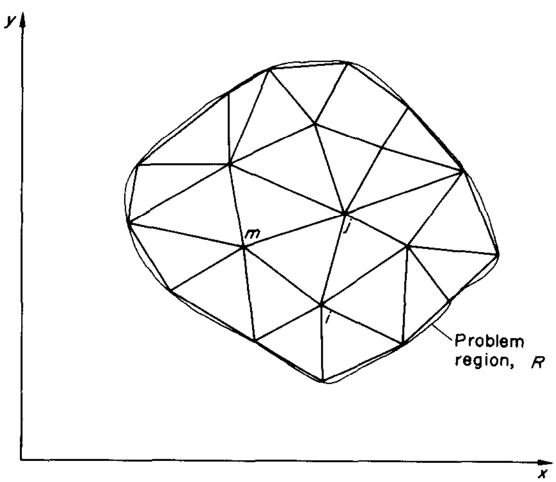

FIG. 2. Division of problem region into "finite elements".

in number to the total number of nodes. Where boundary conditions involve specified nodal values of u at the boundary, say U_k at mode k, the set of simultaneous equations needs to be modified accordingly so as to force $u_k = U_k$. The nodal values of u are then obtained by solving the set of simultaneous equations for u_i.

APPLICATION OF THE FINITE ELEMENT METHOD TO CONVECTION HEAT TRANSFER

(a) *Theory*

The governing equation for the problem considered here is the energy equation with the dissipation term neglected, which may be written in vector form as:

$$\rho C_p\left(\frac{\partial T}{\partial t} + \boldsymbol{v}\,.\,\nabla T\right)\nabla\,.\,(k\nabla T) + Q(\boldsymbol{r}, t) \quad (7)$$

subject to

$$T = T_s \text{ on surface } S_T$$

$$-k\nabla T = \boldsymbol{q} \text{ on surface } S_q. \quad (8)$$

It has been shown [2] that the equivalent minimization functional is

$$I'(T) = \int_V \left[\rho C_p\left(\frac{\partial \bar{T}}{\partial t} + \boldsymbol{v}\,.\,\nabla\bar{T}\right)T + \frac{k}{2}(\nabla T)^2 - QT\right]\mathrm{d}V + \int_{S_q}(\boldsymbol{q}\,.\,\boldsymbol{n})T\,\mathrm{d}S. \quad (9)$$

$\bar{T}$ is the stationary value of T and, hence, not subject to variation. It is the solution of the given problem.

For the situation described in Fig. 1 the energy equation reduces to

$$\rho C_p u\frac{\partial T}{\partial x} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] \quad (10)$$

and the equivalent minimization functional to

$$I'(T) = \int_A \left[\rho C_p u\frac{\partial \bar{T}}{\partial x}T + \frac{k}{2}\left\{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right\}\right] \mathrm{d}A + \int_{S_q}(\boldsymbol{q}\,.\,\boldsymbol{n})T\,\mathrm{d}S. \quad (11)$$

The following set of dimensionless variables is introduced:

$$X = 2x/D_h Pe; \qquad Y = y/R;$$

$$u^+ = u/u_{\max} = 1 - Y^2; \qquad \theta = \frac{T - T_e}{\varepsilon}$$

where $\varepsilon = C$ for case (1) with both wall surface temperatures varying linearly, and $\varepsilon = -q_w D_h/k$ for case (2) with a constant heat flux, q_w (which is positive when directed outwards) at the upper wall and insulated at the lower wall. θ as defined above, will always be positive whatever the signs of C and q_w. Non-dimensionalising equations (10) and (11), we obtain;

$$u^+\frac{\partial\theta}{\partial X} = 8\left[\frac{1}{(2Pe)^2}\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2}\right] \quad (12)$$

$$I(\theta) = \frac{4I'(T)}{\varepsilon^2 kPe} = \int\int\left[u^+\frac{\partial\bar{\theta}}{\partial X}\theta + 4\left\{\frac{1}{(2Pe)^2} \times \left(\frac{\partial\theta}{\partial X}\right)^2 + \left(\frac{\partial\theta}{\partial Y}\right)^2\right\}\right]\mathrm{d}X\,\mathrm{d}Y + \frac{4}{k\varepsilon Pe}\int_{S_q}(\boldsymbol{q}\,.\,\boldsymbol{n})\,\theta\,\mathrm{d}S. \quad (13)$$

When triangular elements are used, the unknown function θ may be approximated as varying linearly within the elements. Thus within an element,

$$\theta = \alpha_1 + \alpha_2 X + \alpha_3 Y \quad (14)$$

where α_1, α_2, α_3 are known functions of the nodal values θ_i, θ_j and θ_m. Rearranging equation (14) we may express θ as a function of the nodal values as follows:

$$\theta = N_i\theta_i + N_j\theta_j + N_m\theta_m \quad (15)$$

where

$$N_i = \frac{1}{2\Delta}(a_i + b_i X + c_i Y),$$

$$a_i = X_j Y_m - X_m Y_j,$$

$$b_i = Y_j - Y_m,$$

$$c_i = X_m - X_j, \text{etc.} \quad (16)$$

As defined above, θ is unique and continuous everywhere and its first derivatives are finite at the element boundaries. Hence, equation (5) holds and

$$I = \sum_e I^e.$$

For I to be a minimum with respect to the nodal values of θ,

$$\frac{\partial I}{\partial \theta_i} = \sum_e \frac{\partial I^e}{\partial \theta_i} = 0 \tag{17}$$

for all i.

Now

$$I^e = \int\int_e \left[u^+ \frac{\partial \bar{\theta}}{\partial X} \theta + 4 \left\{ \frac{1}{(2Pe)^2} \left(\frac{\partial \theta}{\partial X} \right)^2 + \left(\frac{\partial \theta}{\partial Y} \right)^2 \right\} \right] \mathrm{d}X\,\mathrm{d}Y + I_s^e \tag{18}$$

where

$$I_s^e = \frac{4}{k\varepsilon Pe} \int_{S_q^e} (\boldsymbol{q} \,.\, \boldsymbol{n}) \theta \,\mathrm{d}S.$$

I_s^e exists only for elements forming the boundary S_q. Hence

$$\frac{\partial I^e}{\partial \theta_i} = \int\int_e \left[u^+ \frac{\partial \bar{\theta}}{\partial X} \cdot \frac{\partial \theta}{\partial \theta_i} + 8 \left\{ \frac{1}{(2Pe)^2} \frac{\partial \theta}{\partial X} \cdot \frac{\partial}{\partial \theta_i} \times \left(\frac{\partial \theta}{\partial X} \right) + \frac{\partial \theta}{\partial Y} \cdot \frac{\partial}{\partial \theta_i} \left(\frac{\partial \theta}{\partial Y} \right) \right\} \right] \mathrm{d}X\,\mathrm{d}Y + \frac{\partial I_s^e}{\partial \theta_i}$$

$$= \int\int_e [(1 - Y^2)(a_i + b_i X + c_i Y) \times (b_i\theta_i + b_j\theta_j + b_m\theta_m)/(4\Delta^2)]\,\mathrm{d}X\,\mathrm{d}Y + \frac{2}{\Delta}\left[\frac{b_i}{(2Pe)^2} (b_i\theta_i + b_j\theta_j + b_m\theta_m) + c_i(c_i\theta_i + c_j\theta_j + c_m\theta_m) \right] + \frac{\partial I_s^e}{\partial \theta_i}.$$

Letting

$$J_i = \iint_e (1 - Y^2)(a_i + b_i X + c_i Y)\,\mathrm{d}X\,\mathrm{d}Y,$$

$$\frac{\partial I^e}{\partial \theta_i} = \sum_j h_{ij}\theta_j + \frac{\partial I_s^e}{\partial \theta_i}$$

where

$$h_{ij} = \frac{J_i b_j}{4\Delta^2} + \frac{b_i b_j}{2\Delta Pe^2} + \frac{2c_i c_j}{\Delta}. \tag{19}$$

For case (2) where a non-zero heat flux boundary condition occurs along the upper wall surface,

$$I_s^e = \frac{4}{k\varepsilon Pe} \int_{S_q^e} q_w \theta \,\mathrm{d}x = -2 \int_{S_q^e} \theta \,\mathrm{d}X. \tag{20}$$

For an element (i, j, m) whose side im forms part of the boundary surface S_q (see Fig. 3), the

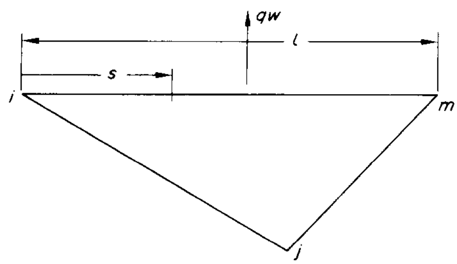

FIG. 3. Element with heat flux on surface *im*.

variation of θ along the side im is given by

$$\theta = \theta_i + (s/l)(\theta_m - \theta_i). \tag{21}$$

Hence, from equation (20)

$$I_s^e = -2 \int_0^l \left\{ \theta_i + \frac{s}{l}(\theta_m - \theta_i) \right\} \mathrm{d}s$$

$$= -l(\theta_i + \theta_m).$$

Hence

$$\frac{\partial I_s^e}{\partial \theta_i} = -l.$$

In general,

$$\frac{\partial I_s^e}{\partial \theta_i} = -l_f^e \,\delta_{fi}$$

where f is a node on S_q, l_f^e is the length of the side of the particular triangular element e forming part of S_q,

$$\text{and } \delta_{fi} = \begin{cases} 1 \text{ if } f = i \\ 0 \text{ if } f \neq i. \end{cases}$$

Hence the elemental contribution $\{\partial I/\partial\theta\}^e$ to the differential $\partial I/\partial\theta$ may be summarised as follows:

$$\left\{\frac{\partial I}{\partial \theta}\right\}^e = \begin{Bmatrix} \dfrac{\partial I^e}{\partial \theta_i} \\ \dfrac{\partial I^e}{\partial \theta_j} \\ \dfrac{\partial I^e}{\partial \theta_m} \end{Bmatrix} = [h]^e \{\theta\}^e + \left\{\frac{\partial I_s}{\partial \theta}\right\}^e \tag{22}$$

where

$$\{\theta\}^e = \begin{Bmatrix} \theta_i \\ \theta_j \\ \theta_m \end{Bmatrix}.$$

In accordance with equation (17), summing the contributions from all the elements in the region to $\partial I/\partial\theta_i$ and equating this sum to zero will yield an equation for the node i. Thus a set of linear algebraic equations involving the nodal values of θ as unknowns is obtained which may be summarised as follows:

$$[h]\{\theta\} + \{P\} = 0. \tag{23}$$

Equation (23) is then modified for boundary conditions involving specified values of temperature. The solution of equation (23) will give the required values of θ at the nodes.

(b) *Computational methods*

The problem region is divided up into a mesh of triangular "finite elements" illustrated in Fig. 4. The elements are smaller near the wall and the entrance since it is expected that temperature gradients will be greater in these regions. The ease with which the size of the elements can be graded in this way to accommodate large temperature gradients is one of the outstanding features of the finite element method.

For case (1), in which the temperature at both wall surfaces varies linearly, the symmetry of the problem permits that only the upper half of the region need be considered. The length of the mesh is chosen to be just large enough that the

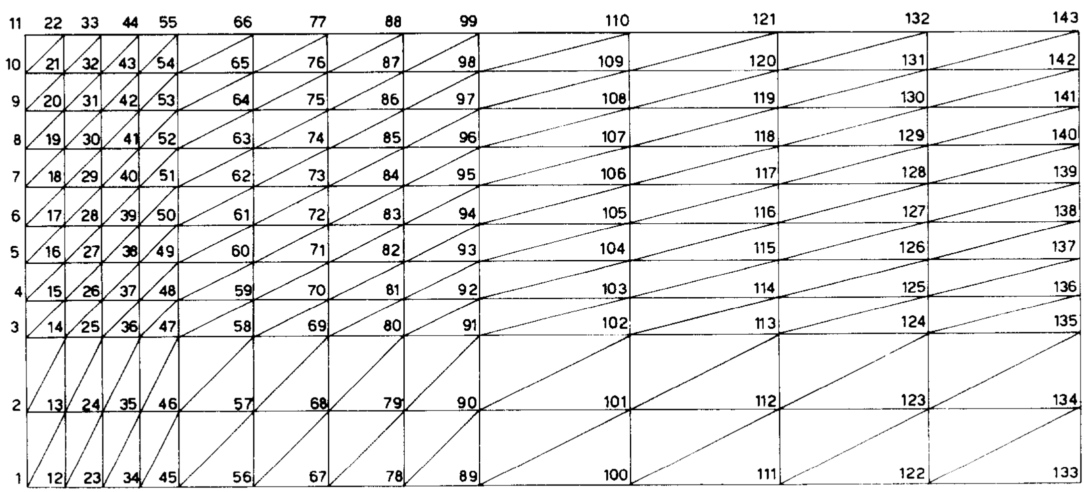

FIG. 4. A typical 13 × 11 mesh.

right hand boundary is in the thermally developed region, for which the temperature profile is known.

The fully developed temperature profile for case (1) is

$$\theta = X - \frac{1}{96}(Y^4 - 6Y^2 + 5)$$

and for case (2), with one wall at constant heat flux and the other insulated, it is

$$\theta = 1{\cdot}5X + \frac{1}{64}(8Y + 6Y^2 - Y^4) - \frac{39}{2240}.$$

At the entrance section, of course, $\theta = 0$ for all Y. Thus for both cases, the temperature at the left and right hand boundaries of the problem region are known. For case (1), the temperature at the upper boundary is given by the wall temperature, $\theta = X$; at the lower boundary the condition of zero heat flux $k\partial(T - T_e)/\partial y = 0$ is imposed. (It may be noted here that in the case of a boundary with zero heat flux, I_s is zero for this boundary.) For case (2), the boundary conditions at the upper and lower boundaries are $k\,\partial(T - T_e)/\partial y = -q_w$ and $k\,\partial(T - T_e)/\partial y = 0$, respectively.

Owing to the manner in which it is assembled from the contributions of each element, the matrix $[h]$ is banded centrally about the diagonal. Generally, the width of the band will vary from row to row depending on how the region is discretised and nodes numbered. At the ith row, the width of the band will be the difference between the largest and smallest nodal numbers adjoining i. For the problem considered, $[h]$ is unsymmetric.

To solve equation (23) directly by using, say, the Gauss–Jordan elimination procedure would be very inefficient in terms of computer time and storage, since that does not take advantage of the banded nature of $[h]$. It appears that the most efficient procedure is to store those elements of $[h]$ that are within the band row-wise in a single vector, $\{H\}$ and employ a modified Gaussian elimination method with back substitution which takes advantage of the banded nature of $[h]$. In this procedure, Gaussian elimination and back substitution need only be carried out up to the lower and upper edges respectively of the band. Thus the zeros of $[h]$ outside the band are not operated upon and are actually not stored in $\{H\}$. With this method of solution, it will be necessary to know the width

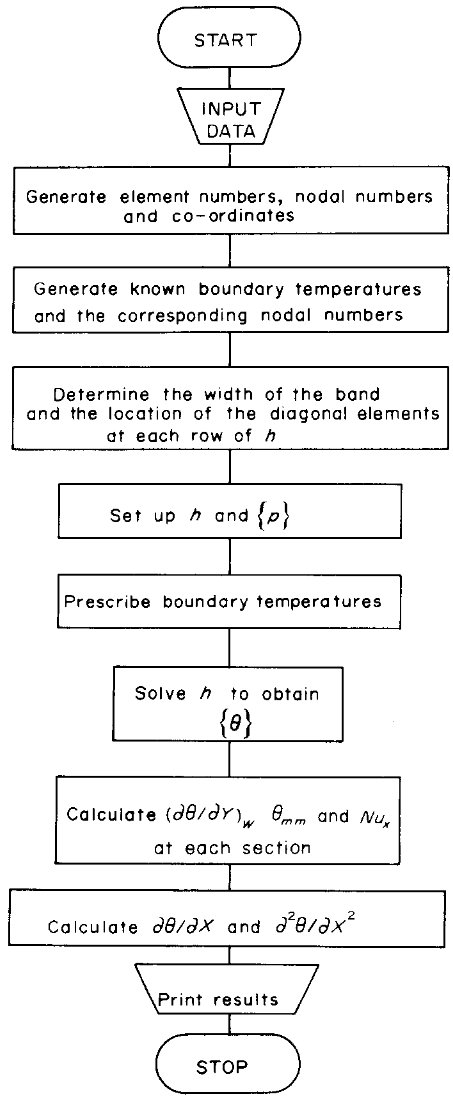

FIG. 5. Flow chart of computer programme.

of the band and the location of the diagonal element within the band at every row of $[h]$.

The various stages in the solution of the convection heat transfer problem by the finite element method are illustrated by the flow chart of Fig. 5.

For calculating the temperature gradients at the wall, two methods are available. The first is to note that within an element

$$\frac{\partial\theta}{\partial Y} = \frac{1}{2\Delta}(c_i\theta_i + c_j\theta_j + c_m\theta_m).$$

Hence we may calculate the values of $\partial\theta/\partial Y$ associated with all the elements adjoining the node at the wall and take the average of these as the temperature gradient $(\partial\theta/\partial Y)_w$ at that point on the wall. This method, however, is not accurate as the approximation will be of the first order only. The alternative method is to employ a finite difference formula; this will be more accurate as a higher order approximation may be employed. For the numerical results presented below, a three-point finite difference formula was used. The same comments apply to the calculation of $\partial\theta/\partial X$ and $\partial^2\theta/\partial X^2$.

The mixed mean temperature at any section is

$$\theta_{mm} = \tfrac{3}{4}\int_{-1}^{1} u^+ \,\theta \, dY.$$

The integration across the section is performed numerically using Simpson's one-third and/or three-eight rules. The local Nusselt number, Nu_x, is calculated from

$$Nu_x = \frac{4(\partial\theta/\partial Y)_w}{\theta_w - \theta_{mm}}.$$

COMPARISON WITH THEORETICAL SOLUTIONS

Theoretical solutions to the problem of convection heat transfer between two parallel planes subject to all the different boundary conditions listed earlier have been obtained [4] with the first term in the bracket of equation (12) neglected. Physically, this is tantamount to neglecting heat conduction in the direction of the flow. For large Péclét number flow, this is obviously justified. In order to make a comparison between numerical and theoretical results, the same term is neglected in the numerical solution, although it could be easily included as long as a Péclét number is specified.

Case (1): *Both wall surface temperatures linearly varying*

Figure 6(a) shows the temperature profiles obtained by the finite element method for this case. These are in accordance with what one would expect for this problem. At small values of X, virtually no heat is transported to the central region, causing the temperature there to remain practically zero. For larger values of X, the temperature profiles gradually become fully developed. The transverse temperature gradient at the centreline is zero as required by symmetry. Figure 6(b) shows that the temperature gradient at the wall starts from zero and rapidly increases to a limiting value of 0·0835; this agrees very well with the theoretical value of 0·0833. The axial variation of the mixed mean temperature

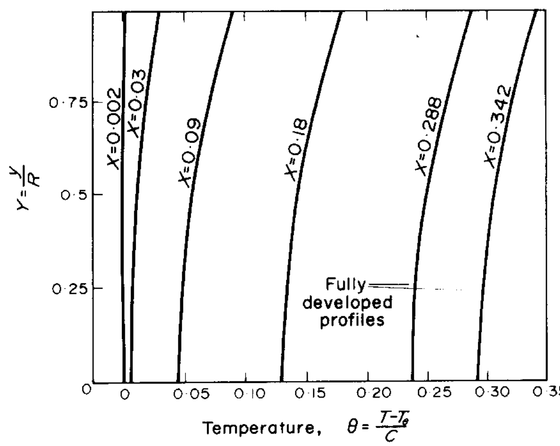

FIG. 6(a). Temperature profiles. Case 1: Both wall temperatures varying linearly. 21 × 25 mesh.

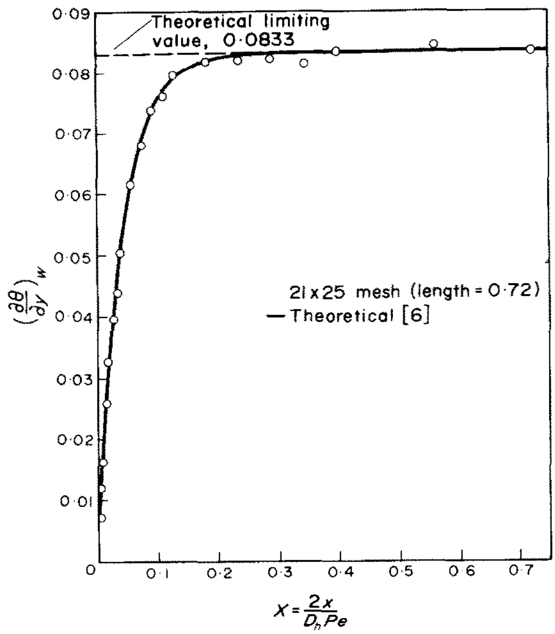

FIG. 6(b). Variation of $\partial\theta/\partial y$. Case 1: Both wall temperatures varying linearly.

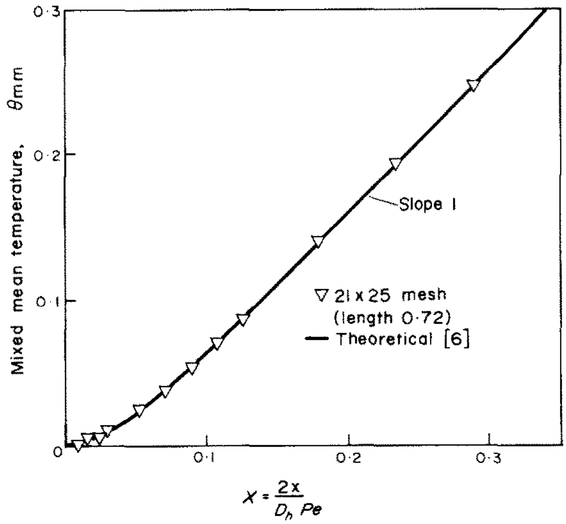

FIG. 6(c). Axial variation of θ mm. Case 1: Both wall temperatures varying linearly.

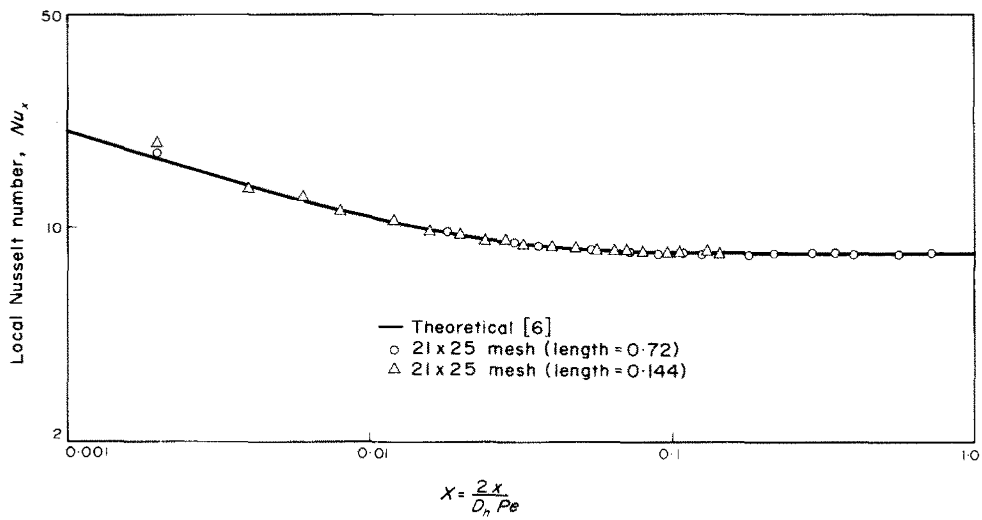

FIG. 6(d). Variation of Nu_x. Case 1: Both wall temperatures varying linearly.

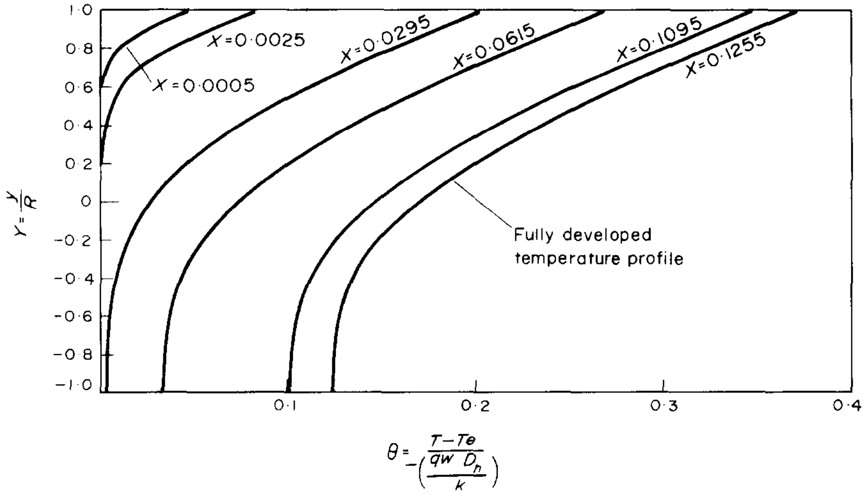

FIG. 7(a). Temperature profiles. Case 2: Upper wall with a constant heat flux, lower wall insulated.

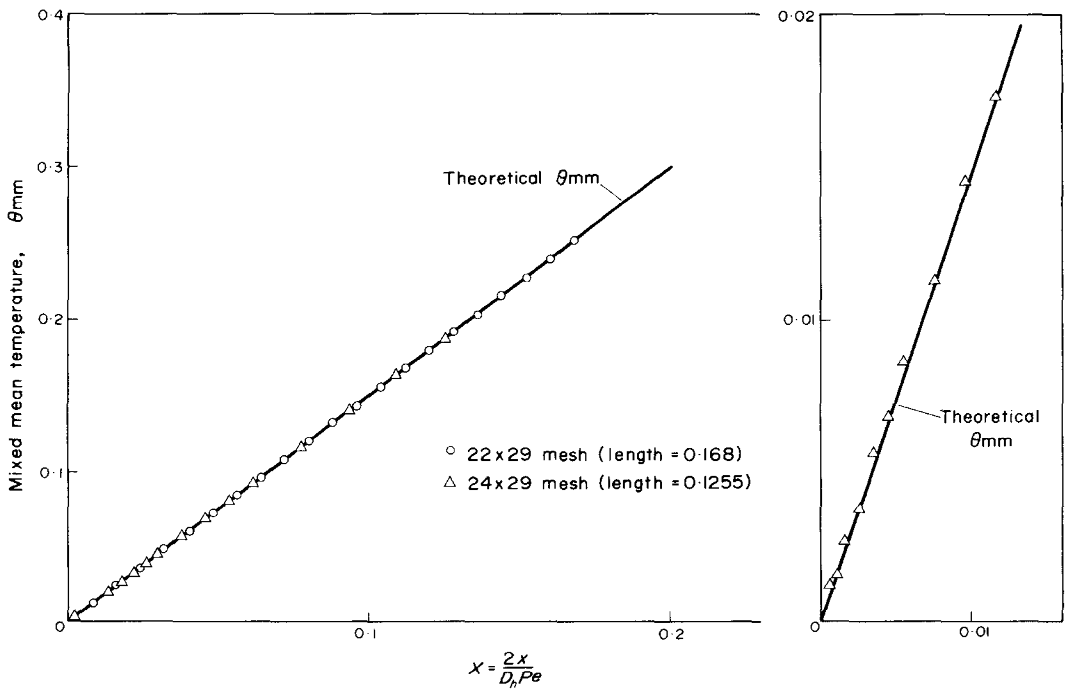

FIG. 7(b). Axial variation of θ mm. Case 2: Upper plate constant heat flux, lower plate insulated.

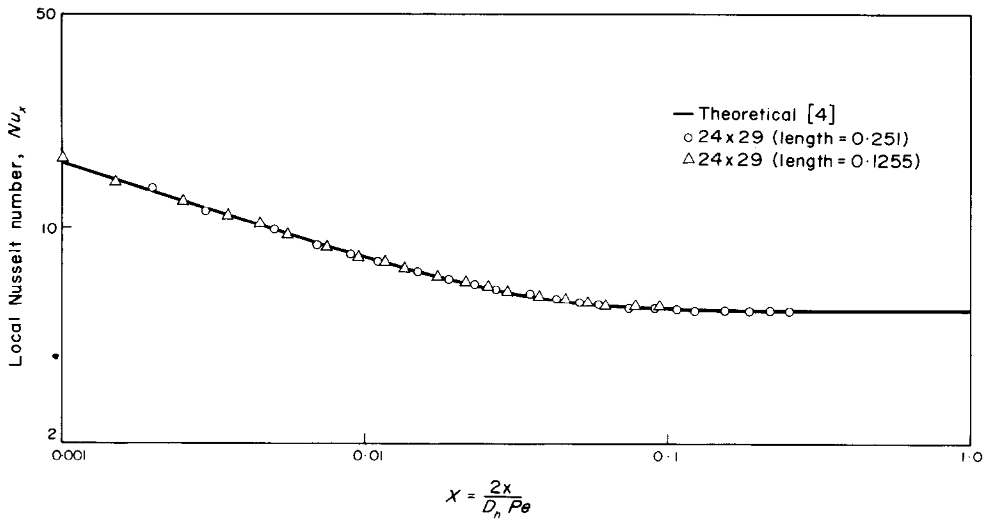

FIG. 7(c). Variation of Nu_x. Case 2: Upper wall with constant heat flux, lower wall insulated.

is plotted and compared with the theoretical solution in Fig. 6(c). The slope of the θ_{mm} vs. X curve tends to a value of 1 as expected, since in the fully developed region $\partial\theta/\partial x = 1$ for all Y.

The computed variation of the local Nusselt number Nu_x in the direction of flow is plotted and compared with the theoretical solution in Fig. 6(d). Agreement between the numerical and theoretical solutions is generally very good except near the entrance. The Nusselt number at the entrance section is of course infinite but rapidly decreases to a limiting value of 8·24. That this occurs in the numerical solution confirms that the temperature profile is fully developed at exit from the solution region. This is further confirmed by the fact that, at exit, the average value across the channel of $\partial\theta/\partial X$ has risen to 1·00 almost exactly and that of $\partial^2\theta/\partial X^2$ has fallen to zero almost exactly.

Case (2): Upper wall with a constant heat flux; lower wall insulated

The temperature profiles obtained by the finite element method are plotted in Fig. 7(a). Again, these are seen to be as expected. For small values of X the temperature is practically zero for the region away from the upper wall. At the lower wall $\partial\theta/\partial Y$ is zero as that wall is insulated. All along the upper wall $\partial\theta/\partial Y$ is equal to the theoretical dimensionless temperature gradient of 0·25 within 1 per cent. The axial variation of the mixed mean temperature is plotted and compared with the theoretical solution $\theta_{mm} = 1{\cdot}5\ X$ in Fig. 7(b). The axial variation of the local Nusselt number of the upper wall obtained numerically is plotted in Fig. 7(c), and compared with the theoretical values from [4].

The agreement between numerical and theoretical solutions is again very good. The local Nusselt number at the upper wall is infinite at the entrance but rapidly decreases to a limiting value of 5·40. The occurrence of a limiting local Nusselt number in the numerical solution coupled with the fact that the average values of $\partial\theta/\partial X$ and $\partial^2\theta/\partial X^2$ across the exit section have reached 1·5 and zero respectively, confirm that the temperature profile at the exit is in fact fully developed.

CONCLUSION

The good agreement between numerical and theoretical solutions verifies the validity of the finite element method. The equivalent minimization functionals for heat and mass transfer and for fluid motion are available [2]. Theoretically therefore, it should be possible to solve numerically any physical problem involving heat or mass transfer or fluid motion by the finite element method.

For the convection heat transfer problem considered, the computational procedure is relatively uncomplicated due to the obvious simplicity of the problem. The set of equations obtained is linear in the nodal values of temperature permitting the use of the well-established Gaussian elimination and back substitution method for its solution. The evaluation of

$$J_i = \iint_e u^+(a_i + b_i X + c_i Y)\,\mathrm{d}X\,\mathrm{d}Y$$

presented no difficulty since u^+ in this case is only a simple function of Y. In other situations where the velocity u is a complicated function of x and y, J_i may have to be evaluated numerically using a quadrature formula which brings with it additional errors.

A further complication arises in many physical problems when the velocity distribution cannot be obtained analytically. Using the appropriate minimization functional, a set of equations for the nodal velocities can be obtained. This is no longer linear, and alternative methods of solution must be adopted.

In this paper, only the three-node triangular element is employed. Other element shapes [3] may be used, but with greater difficulty. The use of the six-node triangular element enables a quadratic approximation of the unknown function within the element to be made; it will however involve the integration of several complicated functions of x and y over the area of the element, which may best be done numerically using a quadrature formula.

Nevertheless, the three-node triangular element seemed to give quite accurate results suggesting that if the sizes of the elements are small enough, a linear approximation of the unknown function within the element is adequate. Owing to the simpler formulation and the ability to cater for arbitrary boundary shapes, the three-node triangular element seems to be adequate for most purposes.

The chief attractions of the finite element method include the ease with which the size and orientation of the elements can be adjusted to cater for regions of high function gradients and arbitrary boundaries respectively. With finite difference methods, on the other hand, variable mesh sizes are used only with considerable complexity and problems with irregular boundaries cannot be readily solved. Moreover, the imposition of flux boundary conditions can be achieved more accurately and easily with finite element methods.

However, the FEM suffers from some drawbacks. Firstly, the truncation error incurred by using a particular element shape cannot be calculated. With the FDM, on the other hand, the truncation error involved in any finite difference formula can be calculated using the calculus of finite differences. However, the order of approximation of a particular element shape is known, and we can say that the truncation error incurred will be comparable with that of a finite difference mesh of the same size.

When a higher order of approximation to the unknown function is sought, the situation usually becomes complex with the FEM, involving difficult area integrations. With the FDM, increasing the order of approximation presents no real difficulty.

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APPLICATION DE LA MÉTHODE AUX ÉLÉMENTS FINIS AU TRANSFERT THERMIQUE PAR CONVECTION ENTRE DEUX PLANS PARALLÈLES

Résumé—On décrit la base de la méthode aux éléments finis appliquée à une équation aux dérivées partielles et la procédure numérique associée. La méthode est alors appliquée au problème du transfert thermique par convection. Le problème consiste en la détermination de la distribution de température et de la variation axiale du nombre de Nusselt local pour un fluide à propriétés physiques constantes qui s'écoule entre deux plans infinis et parallèles dont les températures superficielles et les flux thermiques sont donnés. L'écoulement est hydrodynamiquement développé. La validité de la méthode est verifiée et quelques uns de ses aspects discutés.

ANWENDUNG DES DIFFERENZENVERFAHRENS AUF DEN WÄRMELEITUNGSVORGANG ZWISCHEN EBENEN PLATTEN

Zusammenfassung—Es werden die Grundlagen der zur Lösung von partiellen Differentialgleichungen geeigneten Differenzenmethode und die damit zusammenhängenden numerischen Probleme erörtert. Die Methode wird auf ein Wärmeleitungsproblem angewandt, bei dem die Temperaturverteilung und die axiale Veränderung der lokalen Nusselt-Zahl für eine Flüssigkeit mit konstanten Stoffwerten gesucht sind, welche zwischen zwei unendlich ausgedehnten parallelen Platten strömt, auf denen Oberflächentemperaturen oder Wärmestromdichten vorgeschrieben sind. Die Strömung sei hydrodynamisch ausgebildet. Die Gültigkeit der Methode wird verifiziert und einige Aspekte der Methode werden diskutiert.

ПРИМЕНЕНИЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РЕШЕНИЯ ЗАДАЧИ О КОНВЕКТИВНОМ ТЕПЛООБМЕНЕ МЕЖДУ ПАРАЛЛЕЛЬНЫМИ ПЛОСКОСТЯМИ

Аннотация—Излагаются основы метода конечных элементов применительно к решению дифференциального уравнения в частных производных, а также методика численного решения. Этот метод затем применяется для рещения задачи о конвективном теплообмене. Задача состоит в определении распределения температуры и изменения по оси локального числа Нуссельта для течения жидкости с постоянными свойствами между двумя бесконечными параллельными плоскостями с заданными температурой или тепловыми потоками на поверхности. Течение жидкости полностью развито. Проводится проверка применимости метода, а также обсуждаются его некоторые аспекты.